

ALGEBRAIC CURVES EXERCISE SHEET 3

Unless otherwise specified, k is an algebraically closed field.

Exercise 3.1.

- (1) Show that $V(Y - X^2) \subset \mathbf{A}^2(\mathbf{C})$ is irreducible; in fact, $I(V(Y - X^2)) = (Y - X^2)$.
- (2) Decompose $V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3) \subset \mathbf{A}^2(\mathbf{C})$ into irreducible components.
- (3) Show that $F = Y^2 + X^2(X - 1)^2 \in \mathbf{R}[X, Y]$ is an irreducible polynomial, but $V(F)$ is reducible.

Exercise 3.2.

- (1) Consider the twisted cubic curve $C = \{(t, t^2, t^3); t \in \mathbf{C}\} \subset \mathbf{A}^3(\mathbf{C})$. Show that C is an irreducible closed subset of $\mathbf{A}^3(\mathbf{C})$. Find generators for the ideal $I(C)$.
- (2) Let $V = V(X^2 - YZ, XZ - X) \subset \mathbf{A}^3(\mathbf{C})$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

Exercise 3.3. For topological spaces X and Y , the opens of the product topology on $X \times Y$ are *unions* of products of opens $U \times V$, where $U \subseteq X$ and $V \subseteq Y$. A topological space X is called *Hausdorff* if for any pair of points $x_1 \neq x_2 \in X$, there exist open subsets $U, V \subseteq X$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. A topological space G with an abstract group structure is called a *topological group* if the multiplication and inverse laws are continuous. Let $n \geq 1$.

- (1) Is the product topology on $\mathbb{A}_k^1 \times \mathbb{A}_k^1$ (each copy of \mathbb{A}_k^1 being endowed with the Zariski topology) the same as the Zariski topology on \mathbb{A}_k^2 ?
- (2) Is the Zariski topology on \mathbb{A}_k^n Hausdorff?
- (3) Is $(\mathbb{A}_k^n, +)$ a topological group for the Zariski topology (assuming $\mathbb{A}_k^n \times \mathbb{A}_k^n \simeq \mathbb{A}_k^{2n}$ is endowed with the Zariski topology)?

Exercise 3.4.

- (1) Show that any open subset of an irreducible topological space is irreducible and dense.
- (2) Show that the closure of an irreducible subset of a topological space is irreducible.

Exercise 3.5. Let V an affine variety. Show that algebraic subsets of V are in one-to-one correspondence with radical ideals of $\Gamma(V)$. Show that under this correspondence, affine subvarieties correspond to prime ideals and points correspond to maximal ideals.